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Finite Plane Deformation of a Thick-Walled Cylinder

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A HOLLOW cylinder of internal radius a and external radius b, bonded to a thin shell of thickness h and subject to internal pressure, is considered under the state of plane strain (Fig 1) The material of the cylinder is assumed to be nonlinear elastic solid with strain energy function of arbitrary form

The analysis of stresses and strains is performed under the assumption that the deformation of the cylinder is finite, i e , that the principal extensions, displacements, and displacement gradients are not small in comparison with unity

The general relations of the theory of finite deformations may be found, for example, in Refs 1–4 For the problem dealt with in this note, the cylindrical coordinates r, ϕ, and z are taken as the material coordinates x ($a = 1, 2, 3$):

$$x^1 \equiv r \qquad x^2 \equiv \phi \qquad x^3 \equiv z \tag{1}$$

The spatial coordinates $\bar{x}^k$ ($k = 1, 2, 3$) are

$$\bar{x}^1 = \bar{r} = r + u_r \qquad \bar{x}^2 = \phi \qquad \bar{x}^3 = z \tag{2}$$

where u is the radial displacement

The covariant components of the material strain tensor can be expressed by

$$2\,e_{ab} = (u_{\ |b} + u_{b|a} + u_{\ |a}\, u_{\ |b}) \tag{3}$$

where the vertical bar denotes covariant differentiation, and summation is indicated by repeated indices For the present problem,

$$e_{11} = \frac{(\lambda_1{}^2 - 1)}{2} \qquad e_{22} = r^2\,\frac{(\lambda_2{}^2 - 1)}{2} \qquad e_{33} = 0 \tag{4}$$

where $\lambda_1 = 1 + \partial u_r/\partial r$ and $\lambda_2 = 1 + u/r$ satisfying the compatibility condition

$$d\lambda_2/dr = (1/r)(\lambda_1 - \lambda_2) \tag{5}$$

The strain energy function W is a function of the three invariants of the material strain tensor of the form

$$J_1 = e_a{}^a \qquad J_2 = e_b{}^a e_a{}^b \qquad J_3 = e_b\, e^{\ b} e_a \tag{6}$$

Received November 13, 1963; revision received December 23, 1964 This research was supported by the Office of Naval Research under Contract Nonr 266(78)

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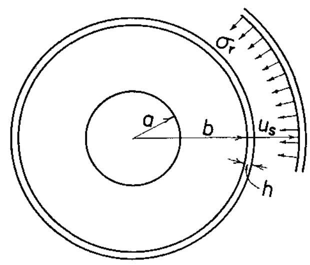

Fig 1 Thick-walled cylinder

The contravariant components of the spatial stress tensor can be expressed by

$$p^{kl} = J^{-1}\,\frac{\partial W}{\partial e_{ab}}\,\frac{\partial \bar{x}^k}{\partial x^a}\,\frac{\partial \bar{x}^l}{\partial x^b} \tag{7}$$

where $J = det\,|\,\delta_b{}^a + u_{\ |b}\,| = \lambda_1\lambda_2$ for the present problem and the deformation gradients $\partial\bar{x}^k/\partial x^a$ follow immediately from Eqs (1) and (2)

The physical components of stress σ_{kl} can be given in terms of p^{kl} as follows:

$$\begin{aligned} \sigma_{11} &\equiv \sigma_r = p^{11} = \lambda_1\lambda_2{}^{-1}W_1 \\ \sigma_{22} &\equiv \sigma_\theta = \bar{r}^2 p^{22} = r^2\lambda_1{}^{-1}\lambda_2 W_2 \\ \sigma_{33} &\equiv \sigma_z = p^{33} = \lambda_1{}^{-1}\lambda_2{}^{-1}W_3 \end{aligned} \tag{8}$$

where the quantities $W_a = \partial W/\partial e_{aa}$ (no summation on a) are equal to the corresponding components of the material stress tensor [1]

The equations of equilibrium in terms of the strain-energy function W are

$$\frac{\partial W}{\partial e_{ab}}\,(\delta_b \ + u_{\ b})_{\ |a} = 0 \tag{9}$$

For the problem under consideration, there is only one nontrivial equation of equilibrium, which is, after some manipulation, obtained from Eq (9) in the form

$$\frac{d}{dr}\,(W_1\lambda_1) + \frac{1}{r}\,W_1\lambda_1 - rW_2\lambda_2 = 0 \tag{10}$$

Performing the differentiation with respect to r and substituting Eq (5), Eq (10) is solved for $d\lambda_1/dr$:

$$\frac{d\lambda_1}{dr} = -\left(\frac{\partial W_1}{\partial\lambda_1}\lambda_1 + W_1\right)^{-1} \times \left[\frac{\partial W_1}{\partial\lambda_2}\,\frac{1}{r}\,(\lambda_1 - \lambda_2)\lambda_1 + \frac{1}{r}\,W_1\lambda_1 - rW_2\lambda_2\right] \tag{11}$$

Equations (5) and (11) represent a system of two differential equations well suited for numerical analysis These equations are easily integrated for λ_1 and λ_2 with the aid of an electronic digital computer The two conditions at $r = b$ which determine the initial values of λ_1 and λ_2 are obtained by prescribing the radial displacement of the shell u and by computing the corresponding pressure on the shell (Fig 1) Since the deformation of the shell is smaller than the deformations in the cylinder and can be assumed to be infinitesimal, the approximate value of the hoop stress σ in the shell is $\sigma = E\,u\,/b(1 - \nu^2)$; hence the pressure p on the intersurface between the shell and the cylinder is

$$p = \frac{E_s}{1 - \nu^2}\,\frac{h}{b}\,\frac{u_s}{b + u} \tag{12}$$

where E and ν are Young modulus and Poisson ratio of the shell

Thus the two conditions at $r = b$ are

$$\lambda_2 = 1 + u\,/b \tag{13}$$

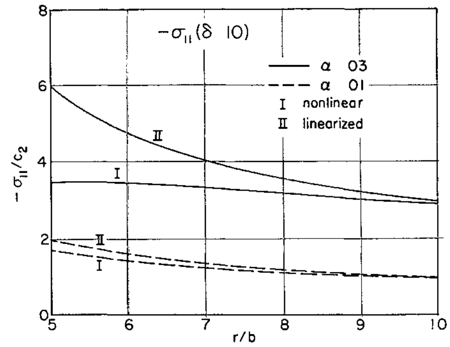

Fig 2 Distribution of radial stress

and, from Eqs (8) and (12),

$$\lambda_1\lambda_2^{-1}W_1 = -\frac{E_s}{1-\nu^2}\frac{h}{b}\frac{u_s}{b+u} \tag{14}$$

In this formulation of the problem, the corresponding magnitude of the internal pressure is obtained by calculating σ_{11} at $r = a$ Therefore, stresses and strains under a specified internal pressure are obtained by trial-and-error method in which Eqs (5) and (11) should be integrated for trial values of u until the computed internal pressure is found to be close enough to the specified value

In the present analysis, it is also possible to include certain nonhomogeneous materials for which the strain-energy function W depends on r explicitly, and not only through λ_1 and λ_2 This can be taken into account during the differentiation with respect to r in Eq (10)

The following form of W has been considered purely for the purpose of illustrating the presented method of analysis; the strain-energy function fails to reproduce certain fundamental physical relations such as the $\lambda_1 - \lambda_2$ relation under simple extension observed in experiments[5]:

$$W = C_1J_1^2 + C_2J_2 \tag{15}$$

The linearized problem corresponding to Eq (15) has also been considered where the Lamé constants $\lambda = 2C_1$ and $u = C_2$ are used for the generalized Hooke's Law with $\epsilon = \partial u_r/\partial r$ and $\epsilon_\theta = u_r/r$

For numerical computations, the following quantities are introduced as parameters:

$$\alpha = \frac{u_s}{b} \qquad \delta = \frac{E_s}{1-\nu}\frac{h/b}{C_2} \qquad \rho = \frac{b}{a} \qquad \gamma = \frac{C_1}{C_2}$$

where $\rho = 2\ 0$, $\gamma = 2\ 0$, $\delta = 10$, and $\alpha = 0\ 01$ or $0\ 03$ are used for numerical integrations The choice of $\gamma = 2\ 0$ implies that the Poisson ratio for infinitesimal deformation is 0 4

The results are shown in Figs 2 and 3 For comparison, the linearized solutions are also plotted in the same diagrams with designation II whereas the curves corresponding to Eq (15) are designated by I

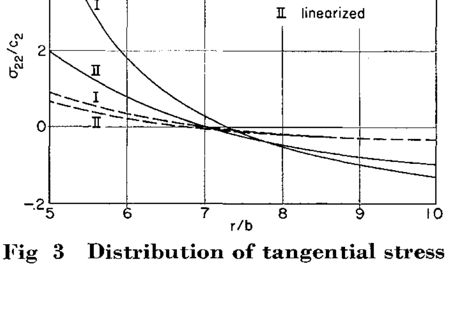

Fig 3 Distribution of tangential stress

The displacement gradients $\partial u/\partial r$ corresponding to Eq (15) are equal to 0 08 at $r = a$ and 0 024 at $r = b$ for $\alpha = 0\ 01$, and 0 33 at $r = a$ and 0 078 at $r = b$ for $\alpha = 0\ 03$

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Effect of Diameter upon Elastic Properties in Thin Fibers

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Introduction

THE use of thin fibers in composite materials is of considerable interest for a variety of applications Examples for the potential value of filamentary composites are the remarkable mechanical properties of filament-wound structures made from endless, thin glass fibers bonded with various organic resins These properties result, in part, from the size effect (increase in strength observed in thin glass fibers as compared to the bulk strength of glass), combined with an effective crack barrier function of the bonding material

Major limitations of those materials are their relatively low elastic modulus and their low temperature resistance

Considerable effort in government-sponsored research and development is expended at the present time to produce fibers of increased stiffness and temperature resistance Principal difficulties encountered in these efforts are 1) the formability of refractory materials is notably poor, and 2) there appears to be little hope for the development of an economical process producing large quantities of nonmetallic fibers of useful length from any other than "glassy" states of materials Glassy states, however, are normally of lower stiffness and temperature resistance than crystalline modifications of the same material Many refractories can be formed into glassy compounds (berillia, borides, etc) If the size effect in those materials can be controlled and exploited, thin fibers made of refractory "glasses" may be expected to provide significant advances in the state of the art of structural materials Thus, the penalties incurred in using glassy states for reasons of formability may be partially or fully compensated by the strengthening effect of small dimensions

A study of the size effects in glass fibers upon elastic (structure-dependent) properties appears worthwhile from two points of view: it may lead to a better understanding of the mechanisms that manifest themselves in the observed size effects on strength and temperature resistance in thin fibers, and the results of such a study may lead to useful guidelines in orienting the search for improved filamentary materials

Received December 13, 1963
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